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ON THE DISCRETE SPECTRUM OF HAMILTONIANS OF MANY-PARTICLES SYSTEMS IN A HOMOGENEOUS MAGNETIC FIELD IN SPACES OF FUNCTIONS GIVEN BY PERMUTATIONAL AND SO (2) SYMMETRY

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ABSTRACT

In this paper we have obtained the conditions of the limiteness and the asymptotics of the discrete spectrum of many-particles hamiltonians with a homogeneous magnetic field. We have investigated the spectrum in the spaces of the given permutational and- especially - SO(2) symmetry. Such an approach permitted to discuss the systems without external potential field and without the supposition that all particles are identical.

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1. In this preprint a new approach is proposed for the investigation of the spectrum n-particles Schrödinger operator with a homogeneous magnetic field. This approach permits us to obtain for the first time the results on the discrete spectrum of Hamiltonians when the external potential field is absent and not all the particles are identical.

Let us discuss the energy operator of a quantum system of particles in a homogeneous magnetic field:

$$\mathcal{H} = -\sum_{j=1}^{n} m_j^{-1} (i\nabla_j + A_j)^2 + \frac{1}{2} \sum_{st;\, s\neq t}^{1,n} V_{st}(|r_{st}|) \ ,$$

where $\nabla_j = \left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial z_j}\right)$, $A_j = \left(-\frac{B}{2} e_j y_j, \frac{B}{2} e_j x_j, 0\right)$, $r_{st} = r_s - r_t$, $r_j = (x_j, y_j, z_j)$ and m_j are the coordinates and mass of j-particle, $\bar{V}_{st}(r_1) = V_{st}(r_1)$, $V_{st}(|r_1|) \in \mathcal{L}_{2,loc}(R^3)$, $\lim_{|r_1'|\to\infty} \int_{|r_1 - r_1'|\leq 1} V_{st}^2(r_1)\, dr_1 = 0$.

After the separation of the centre-of-mass-motion over the direction of the magnetic field vector (the direction of z-axis) we have

$$\mathcal{H} = H - M^{-1} \frac{d^2}{d\zeta_0^2} \ , \qquad (1)$$

where $M = \sum_{j=1}^{n} m_j$, $\zeta_0 = \sum_{j=1}^{n} M^{-1} m_j r_j$ and the operator H is independent of ζ_0.

The equality (1) demonstrates that the study of the spectrum of $\mathcal{H}$ is reduced to the investigation of the operator spectrum H. If all particles are identical, the investigation can be realized after separation of the centre-of-mass motion in the plane x, y. As a result, we shall get the operator H_0, the spectrum of which can be investigated by the method of /1/*. If not all particles are identical, this separation of the centre-of-mass motion is impossible. But without such separation the investigation of the discrete spectrum of the operator H from (1) is practically unreal, because for the operator H, HVZ-theorem may be incorrect. The single result on the spectrum of H in the general situation has been established in /2/, where a procedure of the separation of some variables was suggested which substituted the separation of the centre-of-mass motion in the magnetic field. This method and the result of its application to the operator H - operator H_R - depend essentially on the equality or inequality to zero of the quantity $\sum_{j=1}^{n} e_j$. For the operator H_R in /2/ the version of HVZ-theorem is proved. Other results on the spectrum of H_R are absent.

A completely another approach is proposed by the authors to study the discrete spectrum of H. This approach is based on the utilization of SO(2) symmetry of H. In this preprint it is demonstrated that HVZ-theorem is correct for the restrictions of the operator H on the space $B^{(m)}$ of functions transformed

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* The authors' results on this subject are to be published.

according to the representation of an arbitrary fixed weight m of the group SO(2). Consequently, instead of the procedure of the separation of some variables one can use the procedure of the restriction H on the space $B^{(m)}$. Further we formulate the results of the investigation of the discrete spectrum H on $B^{(m)}$:

i) the reduction of the many-particle problem to two-particle one (theorem 2).

ii) The conditions of the finiteness of the discrete spectrum (theorem 3).

iii) the spectral asymptotics (theorem 4).

All results are obtained with taking into account the permutational symmetry.

2. Let $R^{3n} = \{r\} = \{r_1, \dots, r_n\}$, $R_0 = \{r \mid r \in R^{3n}, \sum_{i=1}^{n} m_i z_i = 0\}$, $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_n)$, $(r, \tilde{r})_1 = \sum_{i=1}^{n} (m_i z_i \tilde{z}_i + x_i \tilde{x}_i + y_i \tilde{y}_i)$, ∇_{03} and Δ_{03} be the gradient and the Laplace operator on R_0. Then under the norm $|r_1| = (r, r)_1^{1/2}$

$$H = \sum_{j=1}^{n} \left[\left(i \frac{\partial}{\partial x_j} - \frac{B}{2} e_j y_j \right)^2 + \left(i \frac{\partial}{\partial y_j} + \frac{B}{2} e_j x_j \right)^2 \right] - \Delta_{03} + \frac{1}{2} \sum_{s,t;\, s \neq t} V_{st}(|r_{st}|).$$

The operator H with the domain $C_0^{\infty}(R_0)$ is essentially self-adjoint in $\mathcal{L}_2(R_0)$. We extend it to a self-adjoint operator saving the previous notation.

Let $Z_2 = (C_1, C_2)$ be the arbitrary decomposition of the initial system $Z_1 = (1, \dots, n)$ into two unempty nonintersecting subsystems C_1, C_2.

$$Q[C_j] = \sum_{i \in C_j} e_i , \qquad Q(Z_2) = Q[C_1]\, Q[C_2] ,$$

$$M(Z_2) = \left[\left(\sum_{i\in C_1} m_i\right)^{-1} + \left(\sum_{i \in C_2} m_i\right)^{-1}\right]^{-1} ,$$

$$R_0[C_j] = \{ r \mid r \in R_0 , \quad r_i = 0 , \quad i \bar{\in} C_j \} ,$$

$$R_0(Z_2) = R_0[C_1] \oplus R_0[C_2] ,$$

$H_0[C_j]$ be the energy operator of subsystem C_j in the field $\mathcal{B} = \{0, 0, B\}$, $H_0[C_j]$ is written after the separation of the center-of-mass motion in the direction of Z -axis;

$H(Z_2) = H[C_1] + H[C_2]$ be the Hamiltonian of the compound system Z_2 ; it is defined on $R_0(Z_2)$.

Let further S , $SO(2)$, W_Z be respectively the groups of the permutations of the identical particles from Z_1 rotations in R^3 around Z -asis, the reflections $Z \leftrightarrow -Z$ in R^3 ; α , m , ω be the types of the irreducible representations $\mathfrak{D}^{\alpha}$, $\mathfrak{D}^{(m)}$ and $\mathfrak{D}_{\omega}$ of these groups by the operators T_g in R_0 , where $T_g \Psi(r) = \Psi(g^{-1} r)$, $g \in S$, $g \in SO(2)$, $g \in W_Z$. We denote by P^{α}, $P^{(m)}$ and P_{ω} the projectors in $\mathcal{L}_2(R_0)$ on the subspaces of functions transforming by operators according to the representations of the types of α , m , ω , respectively.

Let $\sigma = (\alpha, m)$ or $\sigma = (\alpha, m, \omega)$, $H^{\sigma} = H P^{\alpha} P^{(m)}$ if $\sigma = (\alpha, m)$, and $H^{\sigma} = H P^{\alpha} P^{(m)} P_{\omega}$ if $\sigma = (\alpha, m, \omega)$. The purpose of our work is the investigation of the discrete spectrum of the operator H^{σ}.

Let $Z_2 = (C_1, C_2)$, $S[C_j]$ be the group of all permutations of identical particles of the subsystem C_j. Let $\hat{S}(Z_2)$ be the group generated by the group $S[C_1] \times S[C_2]$ and by the permutation $C_1 \leftrightarrow C_2$ if the subsystems C_1, C_2 are identical $(C_1 \sim C_2)$; and let $\hat{\alpha}$, $\hat{m}$, $\hat{\omega}$ be the types of irreducible representations of the groups $\hat{S}(Z)$, $SO(2)$ and W_Z by the operators T_g in $\mathcal{L}_2(R_0(Z))$. When $C_1 \sim C_2$, we denote by $\alpha^-_{\hat{C}}$ the type of such a one-dimensional representation of the group $\hat{S}$, when the number -1 corresponds to the permutation $C_1 \leftrightarrow C_2$, and the number 1 corresponds to all permutations $g \in S[C_1] \times S[C_2]$. Let $-\hat{\alpha}$ be the type of (irreducible) representation of the group $\hat{S}$, which is the direct product of representations $\hat{\alpha}$ and α^-_C; if $C_1 \not\sim C_2$ we write $-\hat{\alpha} = \hat{\alpha}$.

Let $m^{\alpha}_{\hat{\alpha}}$ be the multiplicity of the representation $\hat{\alpha}$ in the representation α after the restriction of the latter from S to $\hat{S}(Z_2)$.

We write further

$$(\hat{\alpha}, \hat{m}) \prec (\alpha, m) \quad \text{if} \quad m^{\alpha}_{\hat{\alpha}} + m^{\alpha}_{-\hat{\alpha}} \geq 1, \ \hat{m} = m,$$

$$(\hat{\alpha}, \hat{m}, \hat{\omega}) \prec (\alpha, m, \omega) \ \text{if} \ m^{\alpha}_{\hat{\alpha}} \geq 1, \ \hat{m} = m, \ \hat{\omega} = \omega$$

$$\text{or} \quad m^{\alpha}_{-\hat{\alpha}} \geq 1, \ \hat{m} = m, \ \hat{\omega} = -\omega.$$

We set $\hat{\sigma} = (\hat{\alpha}, \hat{m})$ if $\sigma = (\alpha, m)$, $\hat{\sigma} = (\hat{\alpha}, \hat{m}, \hat{\omega})$, when $\sigma = (\alpha, m, \omega)$, $P^{\hat{\sigma}}$ is the projector in $\mathcal{L}_2(R_0(Z_2))$ on the subspace of functions transforming by the operators T_g according to the representations of the type $\hat{\sigma}$. Let $H^{\hat{\sigma}}(Z_2) = H(Z_2)P^{\hat{\sigma}}$,

$$H(\sigma, Z_2) = \sum_{\hat{\sigma}(Z_2) \prec \sigma} H^{\hat{\sigma}}(Z_2), \quad \mu^{\sigma} = \min_{Z_2} \inf H(\sigma; Z_2).$$

Theorem 1 (HVZ). The essential spectrum of the operator H^6 consists of all points of a half-line $[\mu^6, +\infty)$.

4. We suppose further that $V_{ij}(|r_1|) = e_i e_j |r_1|^{-\gamma}$ if $|r_1| > C$ for some $C > 0$, $\gamma > 0$. Let $O(6) = \{ Z_2 | \inf H(6; Z_2) = \mu^6 \}$. Everywhere in this preprint we suppose the operator $H(6; Z_2)$ has unempty discrete spectrum for any $Z_2 \in O(6)$.

We denote the decompositions Z', Z'' by the equivalent ones if Z' can be constructed from Z'' by permutations of the identical particles. Then the set $O(6)$ is breaking into the classes $O_i(6)$, $i = 1, 2, \dots, K_0$ of the equivalent decompositions. Choose arbitrary $Z_{2i} = (C_{1i}, C_{2i}) \in O_i(6)$ and denote by $\mathcal{W}(6, Z_{2i})$ the eigenspace of the operator $H(6, Z_{2i})$, which corresponds to the number μ^6.

Let for every class $O_i(6)$

$6_{ij} = 6_{ij}(Z_{2i}) = (\alpha_{ij}, m_{ij}, \omega_{ij})$, $j = 1, \dots, j_i$ be all the types of the symmetry for which $P^{6_{ij}} \mathcal{W}(6, Z_{2i}) \not\equiv 0$, $æ_{ij}$ be the multiplicity of the representation 6_{ij} in $\mathcal{W}(6, Z_{2i})$;

$$d_i(6) = \frac{1}{2} \sum_j æ_{ij} (m^{\alpha}_{\alpha_{ij}} + m^{\alpha}_{-\alpha_{ij}}) \text{ if } 6 = (\alpha, m),$$

$$d_i(6) = \frac{1}{2} \sum_j æ_{ij}\, m^{\alpha}_{\tilde{\alpha}_{ij}} \text{ if } 6 = (\alpha, m, \omega) \text{ here } \tilde{\alpha} = \alpha_{ij},$$

when $\omega_{ij} = \omega$, $\tilde{\alpha}_{ij} = -\alpha_{ij}$ for $\omega_{ij} = -\omega$,

$V_i^{1(2)}(\zeta) = Q(Z_{2i}) |\zeta|^{-\gamma} (\pm) b_1 |\zeta|^{-\gamma - a_i}$, if $|\zeta| \geq b_2$,

$V_i(\zeta) \equiv 0$ if $|\zeta| < b_2$, where b_i are some constants;

$a_i = 2$, when A) or B) is correct:

A) All functions from $\mathcal{W}(6, Z_{2i})$ with the same type of the permutational symmetry have the same parity;

B) $Q[C_{1i}] = Q[C_{2i}] = 0$;

$a_i = 1$ for all other cases.

We denote by $h_i(V^j)$ the operator $-M(Z_{2i})\frac{d^2}{d\zeta^2}+V_i^{1(2)}(\zeta)$, $\zeta\in R^3$ by $N_i(\lambda, V_i^j)$ and $N^\sigma(\mu^\sigma+\lambda)$ the numbers of the eigenvalues with account of multiplicity of the operators $h_i(V_i^j)$ and H^σ which are less or equal, respectively than λ and $\mu^\sigma+\lambda$, $\lambda < 0$.

<u>Theorem 2.</u> Let $\sigma=(\alpha,m)$ or $\sigma=(\alpha,m,\omega)$. Then for some $C>0$

$$-C+\sum_{i=1}^{K_0} d_i(\sigma)N_i(\lambda,V_i^{(1)}) \le N^\sigma(\mu^\sigma+\lambda) \le \sum_{i=1}^{K_0} d_i(\sigma)N_i(\lambda,V_i^{(2)})+C.$$

5. Let us use theorems 1,2 and apply the known results /4/ to the spectrum of one-dimensional operators. Then we obtain the assertions of the finiteness of the discrete spectrum and spectral asymptotics for the operator H^σ.

Let $\sigma=(\alpha,m)$ or $\sigma=(\alpha,m,\omega)$ be arbitrary types of the symmetry.

<u>Theorem 3.</u> The discrete spectrum of the operator H^σ is finite if $\gamma>2$ or if for every Z_{2i}, $i=1,2,\dots,K_0$ at least one of the next conditions is fulfilled 1) - 4):

1) $Q(Z_{2i})>0$, 2) $Q(Z_{2i})=0$ and $\gamma>1$, 3) $Q(Z_{2i})=0$ and for the decompsition Z_{2i} the requirement A) or B) sec 4 is valid, 4) $Q(Z_{2i})>-\frac{1}{4}M(Z_{2i})$ and $\gamma=2$.

<u>Theorem 4.</u> Let $\gamma>0$ and $Q(Z_{2i})<0$ at any rate for one number i, $1\le i\le K_0$. Then for small $|\lambda|$

$$N^\sigma(\mu^\sigma+\lambda)=|\lambda|^{(\gamma-2)/2\gamma}\sum_{i,Q(Z_{2i})<0} d_i(\sigma)M(Z_{2i})^{1/2}|Q(Z_{2i})|^{1/\gamma}J+\mathcal{R},$$

where $\mathcal{R}=O(\ln|\lambda|)$ if $\gamma\geq 1$ or if for all Z_{2i} for which $Q(Z_{2i})<0$ the requirements A) are fulfilled, $\mathcal{R}=O(|\lambda|^{(\gamma-1)/2\gamma})$ for all other cases,

$$J=\gamma^{-1}\int_1^\infty (u-1)^{1/2}u^{-1-\gamma^{-1}}du.$$

Let us note that from theorem 3, the finiteness of the discrete spectrum for Hamiltonians of molecules in the field $\mathcal{B}$ follows, if all $Z_2 \in O(6)$ are decompositions into 2 stable neutral systems; theorem 4 gives the asymptotic of the discrete spectrum specially for the energy operators of atoms, its (+) ions, and of such molecules for which the set $O(6)$ contains the decomposition into two subsystems with changes of the opposite signs.

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