Nizhny Novgorod Radiophysical Research Institute

Ministry of Science, Higher School
and Technological Policy of Russian Federation

P r e p r i n t   N 355

SCHWARZSCHILD PROBLEM IN EINSTEIN GRAVITATIONAL THEORY: AN OPERATIONAL APPROACH

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Nizhny Novgorod 1993

C h e r m y a n i n S. I.

SCHWARZSCHILD PROBLEM IN EINSTEIN GRAVITATIONAL THEORY: AN OPERATIONAL APPROACH // Preprint N 355 - Nizhny Novgorod: Radiophysical Research Institute (NIRFI), 1993. -

For the static Schwarzschild task, the operational principle is introduced as an alternative to the principle of general covariance. According to this principle, the separation between arbitrarily space points has an operational sense, that is, this separation is observed in the context of the given measuring procedure. The content of space operationalism can be specified in various ways. Thus, there exist three types of the operational radial distance for the spherically symmetrical space: the angular distance η expressed in terms of the physical surface $\Sigma = 4\pi\eta^2$, the proper metric length $R = \int g_{rr} dr$ and the locational length $L=c\Delta t$, where Δt is the coordinate time which takes the light signal to travel along the radius L there and back.

The present work studies different types of solutions of the Schwarzschild problem on the basis of the operational assignment of the length. A spherically symmetric static body of a finite size (ideal uncompressable liquid) and its limit case - a massive point-are considered as gravitational sources. Apart from the standard solution (η - solution), we derived a complete R - solution for an extended source. For a point source, besides R-solution, the solution based on the choice of the locational length L (L-solution) was also considered. As for an extended source, we stated the identity between η-solution and R-solution. However, for a massive point, solutions gave the different physical surfaces of this source: zero surface for a standard solution, a surface which coincides with the Schwarzschild singularity surface for R-solution, and a surface in excess of the Schwarzschild one for L-solution. Such a difference is associated with the existence of different limits ($\eta \to 0$, $R \to 0$ and $L \to 0$) from a common extended source to a massive point.

Here, in all cases, as distinct from the general covariant ideology, both the coordinate value of the centre of a spherical massive body and the zero radial coordinate of the point source location are defined before the solution.

"The shutdown" procedure of gravitation has a sense in the context of the operational approach. This procedure correlates Schwarzschild metric with the Minkowsky one, which is to be related to preferred coordinates. It provides for a possibility to unambiguously define such integral constants which can not be found using Newton limit or boundary conditions at infinity from a gravitational source.

1. INTRODUCTION

In the General Theory of Relativity the coordinate systems, used when solving gravitational problem (determination of a metric tensor $g_{\alpha\beta}$ from Einstein's equations), enjoy equal rights and are generally devoid of any physical sense due to the principle of general covariance (see e.g. Møller 1972; Hawking, Ellis 1973; Misner, Thorne, Wheeler 1973). In such a case, the coordinates can take not only positive but negative values as well regardless of the task conditions. Here, the only interval dS has a physical meaning which is an invariant measure of spacing between neighbouring space-time points. However, it seems interesting to consider a distinct problem statement where the preferred coordinate system (accompanying a gravitational source) exists at the expense of the operational principle, which gives a physical content to coordinates. In such a statement, preferred coordinates must immediately represent physical properties of space and time. In the given case, the use of any other coordinate system (in the context of same reference system) will be a simple rearithmetization which doesn't change a physical essence.

We consider that the operational space is such a space where the separation between arbitrarily three-dimensional points is determened using the given measuring procedure. We can have several such operational procedures for each task. Thus, there exist three types of the operational radial distance for the spherically symmetrical space (Schwarzschild problem): the angular distance η expressed in terms of the physical surface $\Sigma = 4\pi\eta^2$, the proper metric length $R = \int g_{rr} dr$ and the locational length $L = c\Delta t$, where Δt is the coordinate time which takes the light signal to travel along the radius L there and back.

The present paper studies different types of solutions of the Schwarzschild problem on the basis of the operational assignment of the length. A spherically symmetric static body of a finite size (ideal incompressable liquid) and its limit case - massive point-are considered as gravitational sources. Apart from the standard solution (η - solution), we derived a complete R - solution for an extended source. For a point source, besides R - solution [Chermyanin, 1992], the solution based on the choice of the locational length L (L - solution) was also considered. As for an extended source, we stated the identity between η - solution and R - solution. However, for a massive point, solutions gave different values for a physical surface of this source: $\Sigma = 0$ for a standard solution, $\Sigma = 4\pi(2GM)^2$ for R - solution, which means that a surface of the source coincides with the Schwarzschild singularity; $\Sigma > 4\pi(2GM)^2$ for L - solution and in this case the metric singularity is absent both in the external space and in the source itself.

Here, in all cases both the coordinate value of the centre of a spherical massive body and the zero radial coordinate of the point source location are defined before the solution. In the general covariant ideology, the point mass on the radial axis is not fixed prior to the solution but is defined from the additional condition $\eta(r) = 0$ as a place of infinite tidal forces. Here, the massive point is determined as a certain central singularity which is not a point limit of a finite source in the ordinary sense.

In the proposed approach there exists the limit $GM \to 0$ that reduces the general gravitational metric to Minkowsky metric written in preferred coordinates.It makes it possible to unambiguously define the additive integral constant C_3 for an external Schwarzschild task (see §§ 2a, 4). From the general covariance standpoint, this limit does not have any sense and the integral constant C_3 can take any value. We can speak about the Minkowsky geometry in this case only at the infinite distance from a gravitational source.

2. INDEPENDENT COMPLETE R-SOLUTION FOR THE SPHERICAL SOURCE OF FINITE SIZE

Consider the operational spherically symmetric space determined by coordinates R, Q, φ. The metric, written in these coordinates, we will denoted as the canonical one. Let static ideal liquid, filling the space inside some sphere with physical radius R_0 is a source of the gravitational field. For simplicity consider the homogeneous liquid, i.e. take μ = const. Then the energy-momentum tensor for the given medium, where macroscopic motion is absent ($u^k = 0$), can be written as

$$T^{\alpha}_{\beta} = (\mu + P)u_{\beta}u^{\alpha} - P\delta^{\alpha}_{\beta} = (\mu + P)\delta_{\alpha 4}\delta^{\beta}_{4} - P\delta^{\alpha}_{\beta}; \qquad (1)$$

where P is pressure, $\delta^{\alpha}_{\beta} = \delta_{\alpha\beta}$ is 4-dimensional Kronecker symbol. For the light velocity, consider $c = 1$.

Let us write the spherically symmetric canonical expression for a space-time interval as follows

$$ds^2 = e^{\nu(R)}dt^2 - dR^2 - \eta^2(R)(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (2)$$

Here ν, η are arbitrary functions on the proper radial variable R, which in its turn is some function on the arbitrary radial coordinate r. The dependence $R = R(r)$ is determined apriori, i.e. before solution is derived. Let $r = R$. Then Einstein equations for the given task are reduced to an independent system

$$\frac{1}{\eta^2} - \frac{\eta'^2}{\eta^2} - 2\,\frac{\eta''}{\eta} = æ\mu, \qquad (3a)$$

$$\frac{\nu'\eta'}{\eta} + \frac{\eta'^2}{\eta^2} - \frac{1}{\eta^2} = æp, \qquad (3b)$$

$$\nu'' + \frac{\nu'^2}{2} + 2\,\frac{\nu'\eta'}{\eta} = æ(\mu + 3p), \qquad (3c)$$

where the stroke marks the differentiation over R, $æ = 8\pi G$,

G is the Newtonian constant. If the boundary of the source is assumed to be free, then the pressure at this boundary is equal to zero and the liquid density, generally speaking, has a continuity break. The condition for hydrostatic equilibrium is written as follows

$$P' = -(\mu + P) \frac{\nu'}{2}. \quad (4)$$

a) External solution.

Outside the source ($\mu = p = 0$), the system (3) has only two independent equations: the equation (3c) is a combination of (3a) and (3b). Rewrite equations (3a), (3b) for the given case as a system

$$2\eta'' - \nu'\eta' = 0, \quad (5a)$$

$$2\eta\eta'' + \eta'^2 - 1 = 0. \quad (5b)$$

The integration of (5a) gives the relation

$$\eta'^2 = c_1 e^{\nu}, \quad (6)$$

where c_1 is integration constant. Equation (5b) can be rewritten in the form

$$(\eta\eta'^2)' = \eta', \quad (7)$$

whence after integration we get the dependence

$$C_1 e^{\nu} = 1 + C_2/\eta, \quad (8)$$

Where C_2 is constant. Taking into account the condition $\eta \to R$ at a large distance from the source and the principle of congruence of the component e^{ν} with the gravity potential $\phi = GM/R$ of the Newton theory, we find the values of constants $C_1 = 1$ and $c_2 = -2GM$, where M is gravitational mass of the source.

A relation between the differentials $d\eta$ and dR is written in the form

$$\frac{d\eta}{\sqrt{1 - 2GM/\eta}} = dR. \quad (9)$$

In the result of integration of (9) we have

$$R = c_3 + \sqrt{\eta - 2GM}\,\sqrt{\eta} + 2GM\ln/\sqrt{\eta/2GM - 1} + \sqrt{\eta/2GM}/, \quad (10)$$

where C_3 is an additive constant. The privileged position of the coordinates R, θ, φ in the operational approach ("R - operationalism") must necessarity imply the existence of the limit $GM \to 0$, which correlates the canonical metric (2) with the Minkowsky one written in same coordinates. Whence, here taking into account $\eta = R$ at $GM = 0$, we get $C_3 = 0$, that corresponds to the Fock's choice of such type of constant [Fock, 1961].

It can seem, that for the determination of C_3 it is necessary to take beforehand into consideration its possible dependence of the reducing mass GM and then the condition $GM = 0$ cannot be used. However, such a dependence exists only for those integration constants (these are C_1 and C_2 in our case), which are determined with the Newtonian limit $g_{00} = 1 - 2\psi$.

Note, that from the general covariance standpoint, the constant C_3 can take any value because such a "shutdown" of gravitation is senseless here. In this case, we can speak about the Minkowsky geometry only at the infinite distance from the source.

b) Inner solution.

Now consider a system (3) inside the spherical source. The equation (3a) can be represented in the form

$$(\eta - \eta\eta'^2)' = \text{æ}\mu\eta^2\eta'. \quad (11)$$

In the result of integration of (11) we have

$$\eta'^2 = 1 - \frac{1}{\eta}(c_4 + \text{æ}\mu\int\eta^2 d\eta), \quad (12)$$

where C_4 is integration constant. Then, taking into account (12) and the relation $p' = \frac{dP}{d\eta}\eta'$, and also the dependence

$$\nu' = \frac{2\eta'' + \text{æ}(\mu + P)\eta}{\eta'},$$

the condition for hydrostatic equilibrium can be written as

$$\frac{dP}{d\eta} = -\frac{\mu + P}{2}\,\frac{C_4 + æ\mu\int\eta^2 d\eta + æp\eta^3}{\eta^2\sqrt{1 - 1/\eta(C_4 + æ\mu\int\eta^2 d\eta)}}. \qquad (13)$$

Using (13), result of summation of equations (3a), (3b) can be expressed as follows

$$(\nu - \ln|\eta'^2|)' = \frac{æ(\mu + P)\eta\eta'}{1 - 1/\eta(C_4 + æ\mu\int\eta^2 d\eta)}. \qquad (14)$$

After the integration of (14), we get the following relation

$$e^{\nu} = C_5\left[1 - 1/\eta(C_4 + æ\mu\int\eta^2 d\eta)\right]\cdot\exp\left\{æ\int\frac{æ(\mu + P)\eta d\eta}{1 - 1/\eta(C_4 + æ\mu\int\eta^2 d\eta)}\right\}, \qquad (15)$$

where C_5 is constant.

The value $\eta_{(0)}$ in the centre of the source is determined by C_4 i.e. in virtue of the integrated form

$$\int_{\eta_{(0)}}^{\eta} \eta^2 d\eta = \int^{\eta} \eta^2 d\eta + C_4. \qquad (16)$$

The choice of $C_4 = 0$ gives us the value of $\eta_{(0)} = 0$. Then integrating (12) we obtain the relation (taking into account that in the spherical centre $R = 0$ as well)

$$\sqrt{\frac{8\pi G}{3}\mu}\; R = \arcsin\left(\sqrt{\frac{8\pi G}{3}\mu}\;\eta\right) \qquad (17a)$$

or an inverse one

$$\sqrt{\frac{8\pi G}{3}\mu}\;\eta = \sin\left(\sqrt{\frac{8\pi G}{3}\mu}\; R\right). \qquad (17b)$$

Integrating (13) and taking into consideration that on the boundary (η_0) of the source the pressure is equal to zero, we get

$$\frac{\mu + 3p}{\mu + p} = \frac{\sqrt{1 - \frac{8\pi G}{3}\mu\eta^2}}{\sqrt{1 - \frac{8\pi G}{3}\mu\eta_0^2}}, \tag{18}$$

from which, taking into account (17b), we have the following formula for the pressure p

$$p = \mu \frac{\cos(\lambda R_0) - \cos(\lambda R)}{\cos(\lambda R) - 3\cos(\lambda R_0)}, \tag{19}$$

where

$$\lambda = \sqrt{\frac{8\pi G}{3}\mu} .$$

The constant C_s is determined from the matching condition on the boundary of the source for a metric component e^{ν}. Using the relation

$$\mu + p = \mu \frac{2\cos(\lambda R_0)}{3\cos(\lambda R_0) - \cos(\lambda R)}$$

in the calculation of the tabular integral

$$\int_0^{\eta} \frac{(\mu + p)\eta d\eta}{1 - \lambda^2\eta^2} = \frac{2\mu}{3\lambda^2} \ln\left|\frac{\cos(\lambda R) - 3\cos(\lambda R_0)}{\cos(\lambda R)(1 - 3\cos(\lambda R_0))}\right|$$

rewrite expression (15) in the form

$$e^{\nu} = C_s\left[\frac{\cos(\lambda R) - 3\cos(\lambda R_0)}{1 - 3\cos(\lambda R_0)}\right]^2 . \tag{20}$$

Writing the matching condition

$$C_s\left[\frac{\cos(\lambda R) - 3\cos(\lambda R_0)}{1 - 3\cos(\lambda R_0)}\right]^2 = 1 - \frac{2GM}{\eta_0(R_0)} \tag{21}$$

and taking into account $\eta_0 = 1/\lambda \sin(\lambda R_0)$ on the boundary of the source, we find the constant

$$C_5 = \frac{[1 - 3\cos(\lambda R_0)]^2}{4}.$$

Then (20) gets the final form

$$e^{\nu} = \frac{1}{4}\,[3\cos(\lambda R_0) - \cos(\lambda R)]^2. \tag{22}$$

Thus the obtained expressions (8), (10), (17), (19) and (22) draw up a complete *R*-solution for a spherical source of a finite size R_0. On the other hand, a standard solution (η - solution) for a source of a finite size η_0 is known

$$ds^2 = (1 - 2GM/\eta)dt^2 - \frac{d\eta^2}{1 - 2GM/\eta} - \eta^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \eta \geq \eta_0$$

(23)

$$ds^2 = \frac{1}{4}\left[\, 3\sqrt{1 - \lambda^2\eta_0^2} - \sqrt{1 - \lambda^2\eta^2}\,\right]^2 dt^2 -$$

$$- \frac{d\eta^2}{1 - \lambda^2\eta^2} - \eta^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \eta \leq \eta_0.$$

But we have learnt already that for R- solution in the source centre $\eta = 0$ as well. Therefore, if a value η_0 of standard solution corresponds to a value R_0 in *R*-solution according to (17) (i.e. the same source is assigned for various solutions), these solutions can be formally considered as one solution fixed in different coordinates and where the coordinate relation is determined by formulae (10) and (17). It means that for the given case the metrics (2) written in arbitrary coordinates and a general η-metrics

$$ds^2 = e^{\nu(\eta)}dt^2 - e^{\lambda(\eta)}d\eta^2 - \eta^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{24}$$

where η is also some arbitrary function on coordinate *r*, are physically equivalent.

3. THE R-SOLUTION FOR A POINT SOURCE

For a massive point, the relations (8) and (10) will be the sought R-solution. The interval (2) can be written as

$$ds^2 = (1 - 2GM/\eta)dt^2 - \frac{d\eta^2}{1-2GM/\eta} - \eta^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (25)$$

where

$$R = \sqrt{\eta - 2GM}\,\sqrt{\eta} + 2GM \ln|\sqrt{\eta/2GM - 1} + \sqrt{\eta/2GM}|.$$

We have $R = 0$ in the place of the source location. However, the value η on the source has here the value $\eta = 2GM$ different from zero. This testifies to the fact that the singularity of this metrics within the open interval $(0,\infty)$ of the proper radial variable R is absent. The metric peculiarity occurs only at $R = 0$. The source possesses here a finite physical surface $\Sigma = 4\pi(2GM)^2$ which coincides with the Schwarzschild sphere [Chermyanin 1992].

4. SINGULARITY-FREE L-SOLUTION FOR A POINT SOURCE

Now let us consider the solution for a massive point on the basis of the choice of "locational" length L. Following the operational ideology, we assume $L = 0$ in the place of source location. Write a spherically symmetric expression for the interval as

$$ds^2 = e^{\nu(L)}(dt^2 - dL^2) - \eta^2(L)(d\theta^2 + \sin^2\theta d\varphi^2), \quad (26)$$

where ν, η are arbitrary functions on the "locational" variable L, which, in its turn, is some function on an arbitrary radial coordinate r. Following a standard procedure, we fix the coordinate system, assuming $r = L$. Then Einstein equations for the given task outside the source are reduced to

$$e^{-\nu}\left(\frac{\nu'\eta'}{\eta} - \frac{\eta'^2}{\eta^2} - 2\frac{\eta''}{\eta}\right) + \frac{1}{\eta^2} = 0, \tag{27a}$$

$$e^{-\nu}\left(\frac{\nu'\eta'}{\eta} + \frac{\eta'^2}{\eta^2}\right) - \frac{1}{\eta^2} = 0, \tag{27b}$$

$$\nu'' + 2\frac{\nu'\eta'}{\eta} = 0, \tag{27c}$$

where the stroke defines the differentiation over L.

The system (27) has only two independent equations - equation (27c) is a combination of (27a) and (27b). We get a system by addition and subtraction of (27a) and (27b)

$$\nu'\eta' = \eta'', \tag{28a}$$

$$e^{\nu} = (\eta\eta')'. \tag{28b}$$

The integration of (28a) gives the relation

$$C_1\eta' = e^{\nu}, \tag{29}$$

where C_1 is integration constant. After the integration of (28b), taking into account (29), we have

$$dr = \frac{d\eta}{C_1 - C_2/\eta}, \tag{30}$$

where C_2 is a constant, and also

$$e^{\nu} = C_1^2 - C_1C_2/\eta. \tag{31}$$

Taking into account boundary conditions at infinity and the principle of the congruence (in analogy with determination of the constant for an external R-solution), we find the values of constants $C_1 = 1$ and $C_2 = +2GM$.

In the result of integration of (29) we have

$$L = \eta - 2GM + 2GM\ \ln|\eta/2GM - 1| + C_3, \tag{32}$$

where C_3 is additive constant. The metrics (26) can be written as follows

$$ds^2 = (1 - 2GM/\eta)(dt^2 - dL^2) - \eta^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (33)$$

Taking into account the privilege of coordinate L in "L-operationalism" and using equality $\eta = L$ at $GM = 0$ we get $C_3 = 0$.

It follows from the relation (32) that η_0 on the source ($L = 0$) has the value of $2GM < \eta_0 < 4GM$ different from zero. This testifies to the fact that within the interval $[0,\infty)$ of the radial variable L the singularity in this metrics is absent. The point source possesses here a finite physical surface $\Sigma = 4\pi\eta_0^2$. At $M \to 0$ the massive point coincides with an ordinary geometrical point.

5. CONCLUSION

The existence of various massive points (as a certain topological formation) having surface different areas, is admissible in the Riemann variety. The choice of the point source will be bound up with the way of determining the operational space. Considering three forms of operational spherically symmetrical space, we have three types of massive points possessing different physical surfaces: $\Sigma = 0$, $\Sigma = 4\pi(2GM)^2$ and $\Sigma > 4\pi(2GM)^2$. But which these sources is more preferable? Following the conviction that rulers and clocks must reflect geometrical properties of any space and time existing in Gravitation, the author thinks that it is more physical to consider the point source of R-solution. The fact that metrical singularity coincides with the surface of the source is more advantageous from the aesthetic standpoint as well.

It conclusion it should be underlined that in the operational approach a standard solution for a point source, where the value of η is a radius-vector of a spherical system, and R-solution (or L-solution) are regarded both as one solution written in different coordinates that is quite reasonable only (unlike the sources of finite physical dimension [Chermyanin 1990]) in the external the space region. Only in this case the expression (10) (or (32)) can be interpreted as a simple coordinate relation between η and R (or L) in the frames of one and the same solution.

ACKNOWLEDGEMENT

The author wishes to express his deep gratitude to Prof. V.S. Troitsky and Prof. S.A.Zhevakin for useful discussions and support in the work. This work was also supported by Grant from the American Astronomical Society.

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