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DESCRIPTION OF THE ESSENTIAL SPECTRUM OF PSEUDORELATIVISTIC HAMILTONIANS, THAT IS INVARIANT WITH RESPECT TO THE PERMUTATIONS OF THE IDENTICAL PARTICLES

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We obtain the location of the essential spectrum of n-particle pseudorelativistic hamiltonians and establish some important properties of the lower bound of this spectrum.

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In this paper we formulate our results on the essential spectrum of many-particle pseudorelativistic hamiltonians without magnetic and external potential fields in the spaces of functions, having arbitrary type α of the permutational symmetry. We discover the location of the essential spectrum for all α and for some cases obtain for the lower bound of this spectrum the formula (15), which is convenient for the study of the discrete spectrum.

Before this time similar results on the essential spectrum were obtain in [2], but in [2] not any α were considered and the construction of the operator of the relative motion was not invariant with respect to the permutations of identical particles in contrast to our approach (in this respect the connection of our results with [2] is the same one, as the connection [4] with [5]); moreover the formula (15) is new.

1. Let $Z_1 = \{0, 1, \ldots, n\}$ be the quantum system of $(n+1)$ particles, m_i, $r_i = (x_i, y_i, z_i)$ and p_i be the mass, the radius-vector and the momentum of i-th particle. Pseudorelativistic (PR) energy operator of Z_1 can be written in the form

$$\mathcal{H}' = K'(r) + V(r), \tag{1}$$

where $r = (r_0, r_1, \ldots, r_n)$,

$$K'(r) = \sum_{j=0}^{n} \sqrt{-\Delta_j + m_j^2}\ ^{*)},$$

$$V(r) = V_0(r) = \frac{1}{2} \sum_{i,j=0,\, i \neq j}^{n} V_{ij}(|r_{ij}|), \tag{2}$$

$\Delta_j = \dfrac{\partial^2}{\partial x_j^2} + \dfrac{\partial^2}{\partial y_j^2} + \dfrac{\partial^2}{\partial z_j^2}$, $V_{ij}(|r_{ij}|) = V_{ji}(|r_{ji}|)$ be the real potential of the interaction i-th and j-th particles, $r_{ij} = r_i - r_j$, $V_{ij}(|r_1|) \in \mathcal{L}_{2,loc}(R^3)$, $V_{ij}(|r_1|) \to 0$ at $|r_1| \to \infty$, and $V_{ij}(|r_{ij}|)$ are

*)We took the unit system in this way, that Plank constant and the light velocity are equal to 1.

such, that for some $\varepsilon_0 > 0$ the operator $\mathcal{H}'$ is semibounded from below for $V(r) = (1+\varepsilon_0)V_0(r)$. If the system Z_1 is a molecule, the last condition means, that we may consider only the molecules, which consist of atoms such elements, whose number in Mendeleev periodical table less than 85 [1, 2].

The operator $\mathcal{H}'$ is not local: in the coordinate space the operators $\sqrt{-\Delta_j + m_j^2}$ are the integral operators, in the momentum representation multiplicators $V_{ij}(|r_{ij}|)$ turn into integral operators. But in the momentum space the operators $\sqrt{-\Delta_j + m_j^2}$ are multiplication operators. Actually let $p_j = (p_{j1}, p_{j2}, p_{j3})$, $p = (p_0, \ldots, p_n)$, $\varphi(r) \in \mathcal{L}_2(R^{3n+3})$, and $\overline{\varphi}(p)$ be Fourie-transform of $\varphi(r)$:

$$\overline{\varphi}(p) = \frac{1}{\left(\sqrt{2\pi}\right)^{3n+3}} \int\limits_{R^{3n+3}} \varphi(r)\, e^{i(p,r)}\, dr,$$

then

$$\sqrt{-\Delta_j + m_j^2}\, \varphi(r) = \sqrt{p_j^2 + m_j^2}\, \overline{\varphi}(p).$$

Let

$$T_j'(p_j) = \sqrt{p_j^2 + m_j^2}, \quad T'(p) = \sum_{j=0}^{n} T_j'(p_j).$$

Now we can rewrite operators $\mathcal{H}'$ using mixed form writing:

$$\mathcal{H}' = T'(p) + V(r), \tag{3}$$

where the operators $T'(p)$ and $V(r)$ act in the momentum and in the coordinate spaces respectively.

2. The operator $\mathcal{H}'$ corresponds to the energy of the whole system motion. But for applications it is interesting to know the spectrum of the operator, corresponding to the *relative* motion energy. To get such operator for nonrelativistic (NR) case one separate the center-of-mass motion, but for pseudorelativistic (PR) case it is impossible. To construct the operator of the relative motion from PR operator $\mathcal{H}'$ we reduce the operator $\mathcal{H}'$ to any fixed eigenspace of

the operator of the total momentum [2]. Let $\xi_0 = (\xi_{01}, \xi_{02}, \xi_{03})$ be the center-of-mass radius-vector:

$$\xi_0 = \sum_{j=0}^{n} m_j r_j / M, \quad M = \sum_{j=0}^{n} m_j,$$

$q_j = r_j - \xi_0$ be the relative coordinates of j-th particle, $j = 0, 1, \ldots, n$, $q = (q_0, \ldots, q_n)$. We take q, ξ_0 as the new coordinates of the particles from Z_1. Let us note, that vectors $q_0, \ldots, q_n$ are dependent: they belong to the space

$$R_0 = \left\{ q' \mid q' = (q'_0, \ldots, q'_n), \quad \sum_{j=0}^{n} m_j q'_j = \theta = (0,0,0) \right\}$$

of the relative motion. From the other hand, if $q' = (q'_0, \ldots, q'_n) \in R_0$ and ξ'_0 is arbitrary fixed vector from R^3, we may consider q'_j and ξ'_0 as the relative coordinates of the point $r'_j = q'_j + \xi'_0$, $j = 0, 1, \ldots, n$ and the center-of-mass position of Z_1 respectively. It is easy to see that Fourie-conjugate coordinates to q_j are the same p_j as for r_j, and Fourie-conjugate coordinate for ξ_0 is $\mathcal{P}_0 = (\mathcal{P}_{01}, \mathcal{P}_{02}, \mathcal{P}_{03}) = \sum_{j=0}^{n} p_j$.

Let us consider the operators

$$L_{0s} = \frac{1}{i} \frac{d}{d\xi_{0s}}, \quad s = 1, 2, 3.$$

In the momentum space these operators hare multiplication operators

$$\overline{L}_{0s} = \mathcal{P}_{0s}.$$

It follows from above that the operators $L_{0s}\{\overline{L}_{0s}\}$ commute with $\mathcal{H}'$. So any eigenspaces of the operators L_{0s} hare invariant for $\mathcal{H}'$. Let $-Q_{0s}$ be a real eigenvalue of the operator L_{0s}, W_{0s} be the corresponding eigenspace and

$$W_0 = W_{01} \cap W_{02} \cap W_{03}.$$

The space W_0 is invariant for $\mathcal{H}'$. Evidently

$$W_0 = \left\{(2\pi)^{-3/2}\, e^{-i(Q_0,\xi_0)}\, \varphi(q)\right\}^{*)},$$

$$\overline{W}_0 = \left\{\overline{\varphi}(p) \prod_{s=1}^{3} \delta(\mathcal{P}_{0s} - Q_{0s})\right\},$$

where $Q_0 = (Q_{01}, Q_{02}, Q_{03})$, $\varphi(q)$ is arbitrary function, $\varphi(q) \in \in \mathcal{L}_2(R_0)$, and $\overline{W}_0$ is Fourie-image of W_0.

Let us rewrite the operator $\mathcal{H}'$ using the coordinates q, ξ_0 $\{p, \mathcal{P}\}$ and reduce it to the subspace $W_0\{\overline{W}_0\}$. Then we obtain the operator $\mathcal{H}'$ in the form

$$H_0' = T'(p, Q_0) + V(q), \tag{4}$$

where

$$T'(p, Q_0) = T'(p),$$

but with the condition

$$\sum_{j=0}^{n} p_j = Q_0; \tag{5}$$

$$V(q) = \frac{1}{2} \sum_{i,j=0, i\neq j}^{n} V_{ij}(|q_i - q_j|), \quad q_i - q_j = r_i - r_j.$$

We see that H_0' depends on the relative coordinates q, their momenta p and the total momentum value Q_0. So if we fix Q_0 we obtain the operator, which can be considered as the operator of the relative motion. We shall study this operator in the space $\mathcal{L}_2(R_0)$ with condition (5) for momenta.

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*)The coefficient $(2\pi)^{-3/2}$ in front of the $e^{-i(Q_0,\xi_0)}$ plays the role of "normalizing factor": Fourie-image of $(2\pi)^{-3/2}\, e^{-i(Q_0,\xi_0)}$ is $\prod_{s=1}^{3} \delta(\mathcal{P}_{0s} - Q_{0s})$ without any factor.

For technical reasons it is convenient to take

$$T_j(p_j) = T'_j(p_j) - m_j$$

instead of $T'_j(p_j)$ and

$$T(p, Q_0) = \sum_{j=0}^{n} T_j(p_j)$$

instead of $T'(p;\, Q_0)$. So the subject of our study is the operator

$$H_0 = T(p;\, Q_0) + V(q) \tag{6}$$

(with condition (5)). The operator H_0 is bounded from below on $C_0^\infty(R_0)$. We extend it to self-adjoint one using Friedrichs extension, and save the notation H_0 for the obtained operator.

Let us note that instead of dependent coordinates $q_0, \ldots, q_n$ we could introduce the independent relative coordinates (and their momenta) similar to [2], but such approach generates the difficulties, when one takes into account the permutational symmetry (see §5) and we do not use this approach.

3. We shall study the spectrum of the operator H_0 not in the whole space $\mathcal{L}_2(R_0)$, but in the subspaces of functions from $\mathcal{L}_2(R_0)$, having the fixed types of permutational symmetry. We do this

i) to satisfy Pauli exclusion principle,

ii) to obtain the additional information about the structure of the spectrum H_0.

We denote by S and α the group of the permutations of all identical particles of Z_1 and the arbitrary type of irreducible representation of S respectively. Let us determine the operators T_g, $g \in S$, by the relations

$$T_g\, \varphi(q) = \varphi(g^{-1}q), \quad g \in S \tag{7}$$

and put

$$P^{(\alpha)} = \frac{l_\alpha}{|S|} \sum_{g \in S} \overline{\chi}_g^{(\alpha)} T_g, \quad B^{(\alpha)} = P^{(\alpha)} \mathcal{L}_2(R_0),$$

where $\chi_g^{(\alpha)}$ be the character of the element $g \in S$ in the irreducible representation of the type α, l_α be the dimension of this representation, $|S|$ be the number of elements of S. The operator $P^{(\alpha)}$ is the projector in $\mathcal{L}_2(R_0)$ on the subspace $B^{(\alpha)} = B^{(\alpha)}(R_0)$ of functions, which are transformed by the operators T_g, $g \in S$, according to the representation of the type α [3]. Evidently $P^{(\alpha)} H_0 = H_0 P^{(\alpha)}$. Let $H_0^{(\alpha)} = H_0 P^{(\alpha)}$. $H_0^{(\alpha)}$ is the restriction of the operator H_0 to the subspace $B^{(\alpha)}$ of the functions, having the permutational symmetry of the type α.

In this paper we discover the location of the essential spectrum $s_{\rm ess}(H_0^{(\alpha)})$ of the operator $H_0^{(\alpha)}$.

4. Let $Z_2 = (D_1, D_2)$ be arbitrary decomposition of the initial system Z_1 into 2 non empty clusters D_1 and D_2 without common elements:

$$D_1 \cup D_2 = Z_1, \quad D_1 \cap D_2 = \emptyset$$

and

$$H(Z_2) = T(p, Q_0) + V(q;\, Z_2), \tag{8}$$

where

$$V(q;\, Z_2) = \frac{1}{2} \sum_{s=1}^{2} \sum_{i,j \in D_s, i \neq j} V_{ij}(|q_j - q_i|).$$

$H(Z_2)$ is PR energy operator of compound system Z_2, consisting of non interacting (one with other) clusters D_1, D_2 with the same condition (5) for total momentum as for Z_1:

$$\sum_{i=0}^{n} p_i = Q_0.$$

Let $S[D_s]$ be the group of the permutations of all identical particles from D_s, $s = 1, 2$, $\hat{g}$ be the permutation $D_1 \leftrightarrow D_2$ if these clusters

are identical ($D_1 \sim D_2$). We put

$$S_0(Z_2) = S[D_1] \times S[D_2],$$
$$S(Z_2) = S_0(Z_2) \text{ if } D_1 \not\sim D_2,$$
$$S(Z_2) = \hat{S}(Z_2) = S_0(Z_2) \cup S_0(Z_2)\hat{g} \text{ if } D_1 \sim D_2.$$

$S(Z_2)$ is the group of the permutational symmetry of the compound system Z_2. It is clear, that $S_0(Z_2) \subseteq S(Z_2) \subseteq S$.

Let $F(\alpha;\, Z_2) = \{\alpha'\}\left\{F_0(\alpha;\, Z_2) = \{\check{\alpha}\}\right\}$ be the set of all types $\alpha'\{\check{\alpha}\}$ of the group $S(Z_2)\{S_0(Z_2)\}$ irreducible representations, which are contained in the group S irreducible representation $D_g^{(\alpha)}$ of the type α after reducing $D_g^{(\alpha)}$ from S to $S(Z_2)\{S_0(Z_2)\}$. For $\forall\alpha'\{\check{\alpha}\}$ we determine the projector $P^{(\alpha')}(Z_2)\{P^{(\check{\alpha})}(Z_2)\}$ on the subspace of functions $\varphi(q)$, which are transformed by the operators T_g

$$T_g\,\varphi(q) = \varphi(g^{-1}q), \quad g \in S(Z_2), \quad \{g \in S_0(Z_2)\}$$

according to the group $S(Z_2)\{S_0(Z_2)\}$ irreducible representation of the type $\alpha'\{\check{\alpha}\}$.

Let $\gamma = \alpha'$ or $\gamma = \check{\alpha}$; obviously if $P^{(\gamma)}(Z_2)\,\varphi(q) = \varphi(q)$, then $P^{(\gamma)}(Z_2)\,\overline{\varphi}(p) = \overline{\varphi}(p)$. We set

$$P(\alpha;\, Z_2) = \sum_{\alpha' \in F(\alpha;\, Z_2)} P^{(\alpha')}(Z_2), \quad \check{P}(\alpha;\, Z_2) = \sum_{\check{\alpha} \in F_0(\alpha;\, Z_2)} P^{(\check{\alpha})}(Z_2),$$
$$H(\alpha;\, Z_2) = H(Z_2)P(\alpha;\, Z_2), \quad \check{H}(\alpha;\, Z_2) = H(Z_2)\,\check{P}(\alpha;\, Z_2).$$

The operator $H(\alpha;\, Z_2)\{\check{H}(\alpha;\, Z_2)\}$ is the restriction of the operator $H(Z_2)$ (see (8)) to the subspace $B(\alpha;\, Z_2) = P(\alpha;\, Z_2)\,\mathcal{L}_2(R_0)$ $\{\check{B}(\alpha;\, Z_2) = \check{P}(\alpha;\, Z_2)\,\mathcal{L}_2(R_0)\}$. Let

$$\mu^{(\alpha)} = \min_{Z_2} \inf H(\alpha;\, Z_2). \tag{9}$$

It is possible to prove that

$$\mu^{(\alpha)} = \min_{Z_2} \inf \check{H}(\alpha;\, Z_2). \tag{10}$$

We denote by $A(\alpha)$ the set of all Z_2, for which

$$\inf \check{H}(\alpha;\, Z_2) = \min_{Z_2'} \inf \check{H}(\alpha;\, Z_2');$$

then

$$\mu^{(\alpha)} = \inf \check{H}(\alpha;\, Z_2), \quad Z_2 \in A(\alpha). \tag{11}$$

5. Our main result is the following theorem

Theorem 1. *Essential spectrum* $s_{\rm ess}(H_0^{(\alpha)})$ *of the operator* $H_0^{(\alpha)}$ *consists of all points half-line* $[\mu^{(\alpha)}, +\infty)$.

Let us compare Theorem 1 with the corresponding results [2].

Firstly, in [2] the similar result was proved only for one of simplest types α of the permutational symmetry (for such α, which corresponds to one-coulomn Young scheme), while we accept arbitrary α.

Secondary, we use more natural, simple and transparent approach for symmetry account as compared with [2]. Actually, we apply the relative coordinates q_i with respect to center-of-mass position ξ_0: $q_i = r_i - \xi_0$, $i = 0, 1, \ldots, n$ and so the transposition g_j: $r_j \leftrightarrow r_0$ of j-th and 0-th particles results in the transposition q_j and q_0 only, but just as all other coordinates q_i, $i \neq j$, $i \neq 0$, are without any changing. In [2] the relative coordinates $\tilde{q}_i$ are taken with respect to the position of 0-th particle: $\tilde{q}_i = r_i - r_0$, $i = 1, 2, \ldots, n$ and this choice implies the chainging of all $\tilde{q}_i$ under transposition g_j. Namely, $T_{g_j}\,\psi(\tilde{q}) = \psi(g_j^{-1}\,\tilde{q}) = \psi(\hat{q})$, where $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n)$, $\hat{q} = (\hat{q}_1, \ldots \hat{q}_n)$, $\hat{q}_i = \tilde{q}_i - \tilde{q}_j$, $i \neq j$, $\hat{q}_j = -\tilde{q}_j$. Such situation is not realized only if the system Z_1 contains a particle, which is not identical to any other particle from Z_1 (and if we index this particle by number 0), but there is no such exceptional particle in the most number of molecules. Completing the second remark we can note roughly speaking that our approach for permutational symmetry account follows [4], while authors [2] follow [5].

6. We do not write here the proof of the Theorem 1, since the significant part of this proof will be needed for the study of the discrete spectrum $s_d(H_0^{(\alpha)})$ of the operator $H_0^{(\alpha)}$ (this study is not

finished), so we shall publish the full proof of the Theorem 1 later (together with the results on the discrete spectrum). But here we shall do some preparations for our next paper. Namely, we shall obtain from (10), (11) the other formula for $\mu^{(\alpha)}$, which is more convenient for the investigation of the structure $s_d(H_0^{(\alpha)})$. To do it first of all we transform the expression of the operator $H(Z_2)$ for fixed $Z_2 = (D_1, D_2)$. We introduce the clusters D_s center-of-mass coordinates

$$\xi_s = (\xi_{s1}, \xi_{s2}, \xi_{s3}) = \sum_{j \in D_s} r_j M_j / M_s, \quad M_s = \sum_{j \in D_s} m_j,$$

the relative coordinates $q_j(Z_2) = r_j - \xi_s$, $j \in D_s$, of the particles from D_s with respect to center-of-mass position and the vector $\eta = \xi_2 - \xi_1$. Evidently, that $q_j(Z_2) = q_j + \xi_0 - \xi_s$, where $\xi_0 - \xi_1 = M_2\eta/M$, $\xi_0 - \xi_2 = -M_1\eta/M$.

The coordinates $q(Z_2) = \Big(q_0(Z_2), \ldots, q_n(Z_2)\Big)$ are not independent, since $\sum\limits_{j \in D_s} m_j\, q_j(Z_2) = \theta$, $s = 1, 2$. It is easy to see that Fourie-conjugate coordinates to $q_j(Z_2)$ be the same p_j, as before. Let $\mathcal{P}_s = \sum\limits_{j \in D_s} p_j$. Then Fourie-conjugate coordinates to η be

$$\mathcal{P}_\eta = (\mathcal{P}_{\eta 1}, \mathcal{P}_{\eta 2}, \mathcal{P}_{\eta 3}) = (\mathcal{P}_2 M_1 - \mathcal{P}_1 M_2)/M \qquad (12a)$$

where by (5)

$$\mathcal{P}_1 + \mathcal{P}_2 = Q_0. \qquad (12b)$$

We consider $q(Z_2)$ and η as new coordinates of particles from Z_1 and denote the operator $H(Z_2)$ in new coordinates by $H_0(Z_2)$. According to above and since $q_i - q_j = q_i(Z_2) - q_j(Z_2)$, $i, j \in D_s$, $s = 1, 2$, we have

$$H(Z_2) = H_0(Z_2) = T(p, Q_0, \mathcal{P}_\eta) + V\Big(q(Z_2);\, Z_2\Big) \qquad (13)$$

where the operator (13) has the same form, as the operator (8), but the conditions (12a, b) have to be hold.

Let us introduce the spaces

$$R_0(Z_2) = \Big\{ q(Z_2) \mid q(Z_2) = \Big(q_0(Z_2), \ldots, q_n(Z_2)\Big), \sum_{j \in D_s} m_j\, q_j(Z_2) = \theta, \quad s = 1,2 \Big\},$$

$$R_\eta = \{\eta \mid \eta = (\eta_1, \eta_2, \eta_3)\}, \quad R_{0,\eta}(Z_2) = R_0(Z_2) \oplus R_\eta,$$

$$\mathcal{L}_2\Big(R_{0,\eta}(Z_2)\Big) = \left\{ \varphi\Big(q(Z_2), \eta\Big) \mid \int\limits_{R_{0,\eta}} |\varphi|^2 dq(Z_2)\, d\eta < +\infty \right\}.$$

In the space $\mathcal{L}_2\Big(R_{0,\eta}(Z_2)\Big)$ we determine the operators $P_0^{(\check{\alpha})}(Z_2)$ similarly to operators $P^{(\check{\alpha})}(Z_2)$, but now the operators T_g, $g \in$ $\in S_0(Z_2)$, are defined on the functions $\varphi\Big(q(Z_2), \eta\Big)$ and $\overline{\varphi}(p, \mathcal{P}_\eta)$ by the relations

$$T_g\, \varphi\Big(q(Z_2), \eta\Big) := \varphi\Big(g^{-1}\, q(Z_2), \eta\Big), T_g\, \overline{\varphi}(p, \mathcal{P}_\eta) = \overline{\varphi}(g^{-1}p, \mathcal{P}_\eta).$$

Here we took into account, that $g^{-1}\eta = \eta$ and $g^{-1}\mathcal{P}_\eta = \mathcal{P}_\eta$ for $\forall \eta, \mathcal{P}_\eta, \forall g \in S_0(Z_2)$.

Let

$$\check{P}_0(\alpha;\, Z_2) = \sum_{\check{\alpha} \in F_0(\alpha,\, Z_2)} P_0^{(\check{\alpha})}(Z_2), \quad \check{H}_0(\alpha;\, Z_2) = H_0(Z_2)\, \check{P}_0(\alpha;\, Z_2).$$

According to (11)

$$\mu^{(\alpha)} = \inf \check{H}_0(\alpha;\, Z_2), \quad Z_2 \in A(\alpha)$$

where the operator $\check{H}_0(\alpha;\, Z_2)$ is considered in the space $\mathcal{L}_2(R_{0,\eta})$. Since the operator $T(p, Q_0, \mathcal{P}_\eta)$ is multiplication operator and the potential $V\Big(q(Z_2);\, Z_2\Big)$ does not depend on η, we may consider the operator $\check{H}_0(\alpha;\, Z_2) \equiv \check{H}_0(\alpha;\, Z_2;\, \mathcal{P}_\eta)$ in the space $\mathcal{L}_2\Big(R_0(Z_2)\Big)$ at the arbitrary fixed $\mathcal{P}_\eta = Q$. Then

$$\mu^{(\alpha)} = \inf_Q \inf \check{H}_0(\alpha;\, Z_2;\, Q), \quad Z_2 \in A(\alpha). \tag{14}$$

The operator $\check{H}(\alpha;\, Z_2;\, Q)$ depends on Q continuously and

$$\lim_{|Q|\to+\infty} \inf \check{H}_0(\alpha;\, Z_2;\, Q) = +\infty$$

since if $|Q| \to +\infty$, then at least for one j it holds $|p_j| \to \infty$ and consequently $T(p, Q_0, Q) \to +\infty$. So there exists a compact set $\Gamma(\alpha;\, Z_2)$ of such vectors $Q \in R^3$ that

$$\mu^{(\alpha)} = \inf \check{H}_0(\alpha;\, Z_2;\, Q), \quad Q \in \Gamma(\alpha;\, Z_2), \quad Z_2 \in A(\alpha).$$

7. Unfortunately in general case we know nothing about finiteness or infiniteness of the number of the set $\Gamma(\alpha;\, Z_2)$ elements. But we can prove the following assertion

Lemma 1. *Let for some open region $W \subset R^3$, $\Gamma(\alpha;\, Z_2) \subset W$,*

i) $\lambda(\alpha;\, Z_2;\, Q) := \inf \check{H}_0(\alpha;\, Z_2;\, Q)$ *is the point of the discrete spectrum of the operator $\check{H}_0(\alpha;\, Z_2;\, Q)$ for $Q \in W$,*

ii) there is such $\check{\alpha}_0$, which does not depend on Q, that the representation $g \to T_g$, $g \in S_0(Z_2)$ in the eigenspace $U(\alpha;\, Z_2;\, Q)$ of the operator $\check{H}_0(\alpha;\, Z_2;\, Q)$, corresponding to its eigenvalue $\lambda(\alpha;\, Z_2;\, Q)$, has ONE irreducible component of the type $\check{\alpha}_0$ for all $Q \in W$.

Then the set $\Gamma(\alpha;\, Z_2)$ is finite.

Remark. The conditions of Lemma 1 forbid the degeneration of the eigenspace $U(\alpha;\, Z_2;\, Q)$ with respect to the symmetry of $S_0(Z_2)$ group only for some type $\check{\alpha}_0$ of the irreducible representation of $S_0(Z_2)$, while for all other types $\check{\alpha}$ the degeneration is possible. It means, that the representation $g \to Tg$, $g \in S_0(Z_2)$, may have in the eigenspace $U(\alpha;\, Z_2;\, Q)$ more than one irreducible component of any type $\check{\alpha}$, $\check{\alpha} \in F_0(\alpha, Z_2)$, $\check{\alpha} \neq \check{\alpha}_0$.

Proof. Let $\check{B}_0(\alpha;\, Z_2) = \check{P}_0(\alpha;\, Z_2)\mathcal{L}_2\Big(R_0(Z_2)\Big)$. Since

$$\check{P}_0(\alpha;\, Z_2) = P_0^{(\check{\alpha}_0)}(Z_2) + \sum_{\check{\alpha}\in F_0(\alpha;\, Z_2),\check{\alpha}\neq\check{\alpha}_0} P_0^{(\check{\alpha})}(Z_2),$$

then

$$B_0^{(\check{\alpha}_0)}(Z_2) := P_0^{(\check{\alpha}_0)}(Z_2)\,\check{B}_0(\alpha;\, Z_2) = P_0^{(\check{\alpha}_0)}(Z_2)\,\mathcal{L}_2\Big(R_0(Z_2)\Big).$$

It follows from the conditions i), ii), that in the space

$$U^{(\check{\alpha}_0)} = U(\alpha;\, Z_2;\, Q) \cap B_0^{(\check{\alpha}_0)}(Z_2) \equiv P_0^{(\check{\alpha}_0)}\, U(\alpha;\, Z_2;\, Q)$$

the representation $g \to T_g$, $g \in S_0(Z_2)$ is irreducible and has the type $\check{\alpha}_0$.

Let $P_{01}^{(\check{\alpha}_0)}$ be the projector in $B_0^{(\check{\alpha}_0)}(Z_2)$ on the space $B_{01}^{(\check{\alpha}_0)}(Z_2)$ functions, which belong to the first line of the group $S_0(Z_2)$ irreducible representation of the type $\check{\alpha}_0$.

Then the space $B_{01}^{(\check{\alpha}_0)}(Z_2)$ is invariant for the operator $H_0(Z_2)$ and in this space the minimal eigenvalue $\lambda(\alpha;\, Z_2;\, Q)$ of the operator $H_0(Z_2)$ is nondegenerated, since the corresponding eigenspace $P_{01}^{(\check{\alpha}_0)}\, U^{(\check{\alpha}_0)}$ is one-dimensional. In other words, the minimal eigenvalue of the operator $P_{01}^{(\check{\alpha}_0)}\, H_0(\alpha;\, Z_2;\, Q)$ is nondegenerated at $\forall Q \in W$. But if $\lambda(\alpha;\, Z_2;\, Q)$ is nondegenerated, then $\lambda(\alpha;\, Z_2;\, Q)$ is analytical function of Q, since the operator $H_0(Z_2)$ is analytical function on Q [6]. That is why there is only finite number of such vectors Q, for which

$$\mu^{(\alpha)} = \lambda(\alpha;\, Z_2;\, Q). \tag{15}$$

Bibliography

1. Lieb E., Yau H.-T. The Stability and Instability of relativistic Matter // CMP. 1988. V. 118. P. 177–213.
2. Lewis R. T., Siedentop H., Vugalter S. The essential spectrum of relativistic multi-particle operators // Ann. Inst. Henri Poincare, Ph. Th. 1997. V. 67. N 1. P. 1–28.
3. Wigner E. P. Group Theory and its Application to the Quantum mechanics, New York, 1959.
4. Sigalov A. G., Sigal I. M. Description of the spectrum of the energy operator of quantum mechanical systems that is invariant with respect to permutations of identical particles. Theor. Math. Phys. Vol. 5, P. 990–1005, 1970
5. Zhislin G. Spectrum of differential operators of quantum mechanical manyparticle system in the spaces of functions of the given symmetry. Izvest. Akad. Nauk SSSR, Ser. Mat. Vol. 33 (1963), N 3, P. 590–649 (English Translation: Math. USSR-Izvestia, Vol. 3 (1969)
6. Reed M., Simon B. Methods of modern mathematical physics. IV Analysis of operators. Academic press, New York – San Francisco – London, 1978.

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